MATHEMATICS

ON THE NON-EXISTENCE OF A CONTINUOUS SELECTOR FOR ARCS LYING IN THE PLANE

BY

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Dedicated to H. Freudenthal on the occasion of his 60th birthday

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It has been recently shown (see [1]) that, if X is a complete separable space, one can assign to each closed non-void subset A of X a point $f(A)$ of A so that the mapping $f: 2^X \to X$, called a *selector*, is of the first class of Baire (which means that the inverse images of open sets are F_σ-sets).[1]

Under some additional assumptions (for example, X being an interval, or a 0-dimensional space), one can sometimes conclude that f is also continuous (compare [2]). It is easy to show that if X denotes the square, a continuous selector does not exist; moreover: there does not exist a continuous selector for the family of all pairs of points $(e^{i\alpha}, e^{i(\alpha+\pi)})$ where $0 \leqslant \alpha < \pi$. The following statement answers a question raised by Professor Morton Brown in this connection.

No continuous selector exists for the subarcs of the square (or even for the set of polygonal lines composed of one or two rectilinear segments).

Proof. Let S_n denote for n odd the segment joining the points $(1/n, 0)$ to $(1/n+1, 1)$; for n even, S_n is the segment with endpoints $(1/n, 1)$ and $(1/n+1, 0)$ (see figure). Put $C = S_1 \cup S_2 \cup \dots$

Let Γ_n be the set of all polygonal lines of the form $S_n \cup A$ where A is a segment contained in C (consequently either $A \subset S_{n+1}$ or $A \subset S_{n-1}$). Put $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots$ and let us denote by S_∞ the interval $[0, 1]$ of the Y-axis. We shall show that

(1) Γ is connected (in the space 2^X where X denotes the unit square),

(2) $\bar{\Gamma} = \Gamma \cup \{S_\infty\}$.

Clearly Γ_n is connected, because any segment A can be reduced in a continuous way to each of its end points, and therefore $S_n \cup A$ can be joined to S_n by an arc lying in the space 2^X. As $(S_n \cup S_{n+1}) \in (\Gamma_n \cap \Gamma_{n+1})$, it follows that Γ is connected.

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1) 2^X denotes the space of all closed non-void subsets of X with the Vietoris topology. If X is compact metric, this topology coincides with the topology induced by the Hausdorff distance between sets.

In order to show that (2) is fulfilled, put

(3) $B = \operatorname*{Lim}_{k=\infty} B_k$ where $B_k \in \Gamma_{n(k)}$, i.e. $B_k = S_{n(k)} \cup A_k$.

First, consider the case where the set $\{n(1), n(2), \ldots\}$ is finite. Then we may assume that $n(k)$ has a constant value n_0 and that

$$B = S_{n_0} \cup \operatorname*{Lim}_{k=\infty} A_k.$$

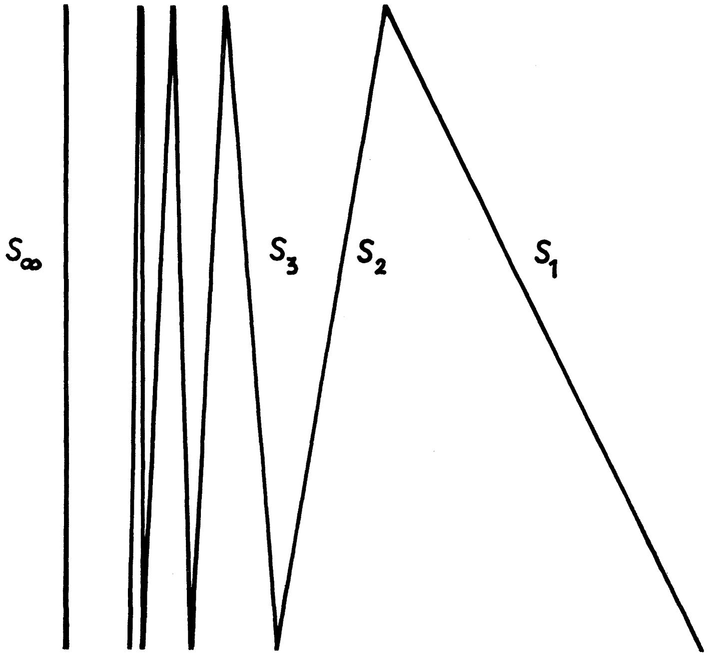
Since $\operatorname{Lim} A_k$ is a segment (or a single point), it follows that $B \in \Gamma_{n_0}$, hence $B \in \Gamma$.

If the set $\{n(1), n(2), \ldots\}$ is infinite, it is legitimate to assume that $\lim_{k=\infty} n(k) = \infty$ and that

$$\operatorname*{Lim}_{n=\infty} B_k = \operatorname*{Lim}_{n=\infty} S_{n(k)} \cup \operatorname*{Lim}_{k=\infty} A_k = S_\infty.$$

This completes the proof of (2).

Suppose now that there is a continuous selector defined on $\bar{\Gamma}$. As $\bar{\Gamma}$ is a continuum so $f(\bar{\Gamma})$ is a continuum as well. Furthermore, it is a subcontinuum of the plane continuum $C \cup S_\infty$ and contains only a single point of S_∞. But this is a contradiction, for each subcontinuum of $C \cup S_\infty$ which contains points of C and of S_∞ obviously contains the whole segment S_∞.



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2. Michael, E., Continuous selections in Banach spaces, Studia Math. 75–76 (1963).